MTH 295
Fall 2019
Homework 5
Due Thursday, 10/10 Name: Key

1) In the case of a repeated root of the equation $ay'' + by' + cy = 0$ we used reduction of order to find a second linearly independent solution. Here is another way to find the second solution.

a) Suppose $e^{k_1 x}$ and $e^{k_2 x}$ are each solutions to the above equation. Show that $\frac{e^{k_1 x} - e^{k_2 x}}{k_1 - k_2}$ is also a solution.

$ay'' + by' + cy = 0$ is linear so if
$y_1 = e^{k_1 x}$ and $y_2 = e^{k_2 x}$ are both solutions
Then $y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$ is a solution
so $y = \frac{1}{k_1 - k_2} e^{k_1 x} - \frac{1}{k_1 - k_2} e^{k_2 x} = \frac{e^{k_1 x} - e^{k_2 x}}{k_1 - k_2}$
is a solution.

b) Treat k_1 as fixed and take the limit of the solution in part a) as $k_2 \to k_1$ to find the second linearly independent solution.

$\lim_{k_2 \to k_1} \frac{e^{k_1 x} - e^{k_2 x}}{k_1 - k_2} = \frac{0}{0}$ so L'Hopital's rule applies

and $\lim_{k_2 \to k_1} \frac{e^{k_1 x} - e^{k_2 x}}{k_1 - k_2} = \lim_{k_2 \to k_1} \frac{-x e^{k_2 x}}{-1}$

$= x e^{k_1 x}$ which is the 2nd independent solution

2) We used reduction of order to find the second solution of the linear homogeneous equation with constant coefficients but the method is more general than that (it always depends on knowing one solution).

a) Show that $y_1 = x$ is a solution of $(1+x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$.

$y_1 = x$
$y_1' = 1$
$y_1'' = 0$

So $(1+x^2)\frac{d^2y_1}{dx^2} - 2x\frac{dy_1}{dx} + 2y_1 = (1+x^2)(0) - 2x(1) + 2x = 0$

and $y_1 = x$ is a solution.

b) Assume that $y_2(x) = y_1(x)v(x) = xv(x)$ is a second solution. Find $y_2(x)$ and write the general solution of the given equation. You should find a first order equation for $p = \frac{dv}{dx}$ which is separable (thus the name "reduction of order").

Suppose $y_2 = y_1 v = xv$

then $y_2' = v + xv'$

$y_2'' = v' + v' + xv''$
$= 2v' + xv''$

and $(1+x^2)y_2'' - 2xy_2' + 2y_2 = (1+x^2)(2v' + xv'') - 2x(v + xv') + 2xv$

$= 2v' + xv'' + \cancel{2x^2v'} + x^3v'' - \cancel{2x^2v'}$

$= 2v' + xv'' + x^3v''$

and if $v' = p$ then

$2p + (x+x^3)p' = 0$ which is separable –

$(x+x^3)\frac{dp}{dx} = -2p$

$-\frac{dp}{2p} = \frac{dx}{x(1+x^2)}$

$-\frac{1}{2}\ln|p| = \int\left\{\frac{1}{x} - \frac{x}{1+x^2}\right\}dx$

$= \ln|x| - \frac{1}{2}\ln|1+x^2| + c_1$

$p = c_1 \cdot \frac{1+x^2}{x^2}$

$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$

$1 = A + Ax^2 + Bx^2 + Cx$

so $C = 0$,
$A = 1$
$B = -1$

and $v = \int p\,dx = \int v'dx = c_1\left(x - \frac{1}{x}\right) + c_2$

so $y_2 = xv = c_1(x^2 - 1) + c_2 x$

and $y = c_3 y_1 + c_4 y_2 = c_3 x + c_4\{c_1x^2 - c_1 - c_2 x\}$

$= \boxed{c_1 x + c_2(x^2 - 1)}$

3) Solve the IVP $\ddot{x}+4\dot{x}+4x=0$, $x(0)=1$, $\dot{x}(0)=3$.

if $x = e^{kt}$

then $k^2+4k+4=0$

$(k+2)(k+2)=0$

$k=-2$ is a repeated root

so. $x = C_1 e^{-2t} + C_2 t e^{-2t}$

$x(0) = C_1 = 1$

$\dot{x} = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$

$\dot{x}(0) = -2C_1 + C_2 = 3$

$C_2 = 3 + 2C_1$

$C_2 = 5$

so $\boxed{x = e^{-2t} + 5te^{-2t}}$

4) Solve the IVP $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 0$.

$y = e^{kx}$

$k^2 - 3k - 4 = 0$

$(k-4)(k+1) = 0$

$k = 4, -1$

$y = c_1 e^{4x} + c_2 e^{-x}$

$y(0) = c_1 + c_2 = 1$

$y' = 4c_1 e^{4x} - c_2 e^{-x}$.

$y'(0) = 4c_1 - c_2 = 0$

so $5c_1 = 1$

$c_1 = \frac{1}{5}$

$c_2 = 4c_1$

$= \frac{4}{5}$

$$\boxed{y = \frac{1}{5} e^{4x} + \frac{4}{5} e^{-x}}$$

5) Solve the IVP $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=0,\ y(0)=1,\ y'(0)=1.$

$y = e^{kx}$

$k^2 + 2k + 4 = 0$

$k = \frac{-2 \pm \sqrt{4-16}}{2}$

$= -1 \pm \sqrt{3}$

$y = e^{-x}\{c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\}$

$y(0) = c_1 = 1$

$y' = -e^{-x}\{c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\} + e^{-x}\{-\sqrt{3}c_1 \sin(\sqrt{3}x) + \sqrt{3}c_2 \cos(\sqrt{3}x)\}$

$y'(0) = -c_1 + \sqrt{3}c_2 = 1$

$c_2 = \frac{1+c_1}{\sqrt{3}} = 2/\sqrt{3}$

$$\boxed{y = e^{-x}\left\{\cos(\sqrt{3}x) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}x)\right\}}$$